On the Coefficient of λ in the Characteristic Polynomial of Singular Graphs

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Abstract

A singular graph, with adjacency matrix A and one zero eigenvalue, has a corresponding eigenvector v_0 which is related to $\mathcal{L}$, the coefficient of λ of the characteristic polynomial $\phi(G, \lambda) =$Det$(\lambda I - A)$. In this paper a simple formula is derived expressing $\mathcal{L}$ in terms of the norm of v_0. Furthermore it is shown that the ratio of the diagonal cofactors, which are the determinants of the adjacency matrices of the vertex-deleted subgraphs of G, can be obtained from a kernel eigenvector. The non-singular vertex-deleted subgraphs of G are characterised. Results are also obtained for singular graphs with more than one zero eigenvalue.

Keywords: nullity, kernel eigenvector, kernel relation, principal kernel eigenvector, core, core-order, periphery, minimal configuration.

1 Introduction

All the graphs we consider are simple, i.e. without multiple edges or loops. If a graph G has n vertices, then the order n is denoted by $n(G)$.

The adjacency matrix $A(G)$ or A of a graph G having vertex set $\mathcal{V}(G) = \{v_1, v_2, \ldots, v_n\}$ is an $n \times n$ **symmetric** matrix (a_{ij}) such that $a_{ij} = 1$ if v_i and v_j are adjacent and 0 otherwise. A is also represented by $(R_1, R_2, \ldots, R_n)^T$ where R_i is the ith row of A corresponding to vertex v_i. The rank of a graph G, $r(G)$ is the rank of its adjacency matrix A. The linear transformation corresponding to A is also denoted by A so that for $u,\ w \in \mathbb{R}^n$, and $\alpha, \beta \in \mathbb{R}$, $A(\alpha u + \beta w) = \alpha A(u) + \beta A(w)$.

A graph is said to be **singular** if its adjacency matrix A is a singular matrix. At least one of the eigenvalues of A is zero. There corresponds a non-zero vector v_0 such that $Av_0 = 0$. The vector v_0 is in the nullspace of A.

Since the adjacency matrix A is symmetric the algebraic multiplicity is the same as the geometric multiplicity for each eigenvalue λ. This common

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value for $\lambda = 0$ is the **nullity** of G, denoted by $\eta(G)$, and is therefore the dimension of the nullspace of A. It follows that $r(G) + \eta(G) = n(G)$.

Definition: *A* **kernel eigenvector** v_0 *of a singular graph with adjacency matrix A, is an eigenvector in the nullspace of A.*

Definition: *Let G be a singular graph having adjacency matrix $A = (R_1, R_2, \ldots, R_n)^T$ and a kernel eigenvector $v_0 = (\alpha_1, \alpha_2, \ldots, \alpha_t, 0, \ldots, 0)^T$, $1 \leq t \leq n$. Then a* **kernel relation** *$\mathcal{R}$ of G (with respect to v_0) is the linear relation*

$$\alpha_1 R_1 + \alpha_2 R_2 + \ldots + \alpha_t R_t = 0. \tag{1}$$

Definition: *A* **principal kernel eigenvector** $v_0 = (\alpha_1, \alpha_2, \ldots, \alpha_t, 0, \ldots, 0)^T$, *$1 \leq t \leq n$, has non-zero integer components $\alpha_1, \alpha_2, \ldots, \alpha_t$ with g.c.d. equal to 1 and $\alpha_1 > 0$.*

Main Theorem : *Let G be a singular graph with adjacency matrix A and nullity one and let G have the principal kernel eigenvector $v_0 = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T$, where $\alpha_i \neq 0$, for $1 \leq i \leq t$, $1 \leq t \leq n$, $\alpha_1 > 0$. Then the squared norm of the principal kernel eigenvector divides $\mathcal{L}$, the coefficient of λ of the characteristic polynomial $\phi(G, \lambda) =$Det$(\lambda I - A)$.*

In related work [4,8], singular graphs of nullity one having the kernel eigenvector with no zero components, called nut graphs, played a central role. The results in this paper present an easily applicable criterion for determining the existence of nut graphs by considering the coefficient of λ in the characteristic polynomial of singular graphs. It was shown that the smallest nut graph is of order 7 and, using this criterion, that there are 3 nut graphs of order 7.

2 The role of the Adjoint

Lemma 1: *Let G be a graph of order n with adjacency matrix $A = (a_{ij})$ and adjoint Adj(A) $= (A_{ij})$, where A_{ij} is the cofactor of a_{ij}. Then each entry of the adjoint is an integer.*

Proof: This follows from the definition of the determinant as a sum of products of n permuted entries of A and the signum of the permutation [5].

Each of the n terms in a product is taken from a different row of A and from a different column of A. As each entry of A is 0 or 1, the determinants expressing the entries of the adjoint are integers.\\

Lemma 2: *Let G be a graph of order n, with adjacency matrix $A = (a_{ij})$ and adjoint $Adj(A) = (A_{ij})$. Then if G is singular, for $i \in \{1, 2, \ldots, n\}$, each row vector of the adjoint, $u_i = (A_{1i}, A_{2i}, \ldots, A_{ni})^T$, is zero or a kernel eigenvector. Furthermore*

i) If $\eta(G) > 1$, then $u_i = 0,\ \forall i \in \{1, 2, \ldots, n\}$.

ii) If $\eta(G) = 1$, then for some $i \in \{1, 2, \ldots, n\}$, u_i is non-zero and a non-zero row vector of the adjoint is unique (up to rational multiples).

iii) If $\eta(G) = 0$, then $r(\mathrm{Adj}(A)) = n$.

Proof: Let $A = (a_{ij}),\ i, j \in \{1, 2, \ldots, n\}$. If the graph is singular, $\mathrm{Det}(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \ldots + a_{in}A_{in} = 0$. Also for $j \neq i,\ a_{j1}A_{i1} + a_{j2}A_{i2} + \ldots + a_{jn}A_{in} = 0$. Since A is symmetric, $R_1A_{i1} + R_2A_{i2} + \ldots + R_nA_{in} = 0$, which can be written as

$$A \begin{pmatrix} A_{i1} \\ A_{i2} \\ A_{i3} \\ \vdots \\ A_{in} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{2}$$

Thus the non-zero row vectors $u_i = (A_{i1}, A_{i2}, \ldots, A_{in})^T$ are kernel eigenvectors.

i) If $\eta(G) > 1$, then $r(A) < n-1$. Thus the determinant of each submatrix of $A(G)$ of order $(n-1)$ is zero. Hence $u_i = 0,\ \forall i \in \{1, 2, \ldots, n\}$.

ii) If $\eta(G) = 1$, then $r(A) = n - 1$. Thus the determinant of at least one submatrix of $A(G)$ of order $(n-1)$ is not zero. Hence $\exists k, l \in \{1, 2, \ldots, n\}$ such that $A_{k,l} \neq 0$. Thus u_k is a non-zero kernel eigenvector of G (and so is u_l). But the dimension of the nullspace (or kernel of $A(G)$) is one. Thus if $\exists j \in \{1, 2, \ldots, n\},\ j \neq k$ such that $u_j \neq 0$, then u_j is also a kernel eigenvector and $u_j = cu_k,\ c \in \mathbb{Q}$. Hence the non-zero $u_i,\ i \in \{1, 2, \ldots, n\}$, are rational multiples of each other and give the unique kernel eigenvector (up to rational multiples).

iii) If $\eta(G) = 0$ then G is a non-singular graph of order n. Hence A^{-1} exists and is non-singular. As $A^{-1} = \dfrac{\text{Adj}(A)}{\text{Det}(A)}$, then $r(\text{Adj}(A)) = r(A^{-1}) = n$. \\

Lemma 3: *Let G be a singular graph with adjacency matrix A.*

i) If $\eta(G) > 1$ then $Adj(A) = 0$.

ii) If $\eta(G) = 1$ then $r(Adj(A)) = 1$.

Proof: Each row vector of the adjoint is zero or a kernel eigenvector of A. From Lemma 2,

i) if $\eta(G) > 1$ then $u_i = 0,\ \forall i \in \{1, 2, \ldots, n\}$. Thus $\text{Adj}(A) = 0$.

ii) If $\eta(G) = 1$ then a row vector of the adjoint of its adjacency matrix either has each entry zero or if it is non-trivial then it is unique (up to rational multiples). Since at least one row vector of the adjoint is non-zero then $r(\text{Adj}(A)) = 1$.\\

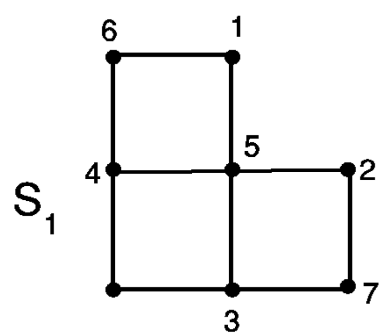

Figure 1:

The graph S_1 has nullity 2 and the adjoint of its adjacency matrix is the zero matrix.

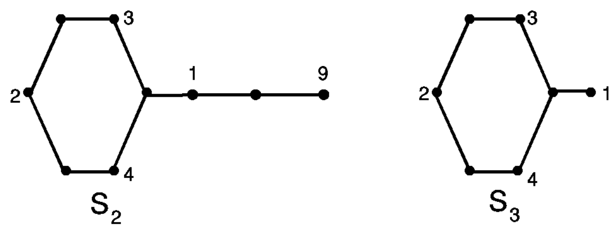

Figure 2:

The graphs S_2 and S_3 are both of nullity one. The adjoints of the corresponding adjacency matrices, are
$$\begin{pmatrix} 4 & 2 & -2 & -2 & 0 & 0 & 0 & 0 & -4 \\ 2 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & -2 \\ -2 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 2 \\ -2 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & -2 & 2 & 2 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$
and
$$\begin{pmatrix} -4 & -2 & 2 & 2 & 0 & 0 & 0 \\ -2 & -1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & -1 & -1 & 0 & 0 & 0 \\ 2 & 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$
respectively. Both matrices have rank one.

Corollary: *Let G be a singular graph. Then $r(Adj(A)) = 1$ if and only if $\eta(G) = 1$.*

Proof: The condition is sufficient by Lemma 3. To prove that it is necessary, suppose $\eta(G) > 1$. Then by Lemma 3, $\mathrm{Adj}(A) = 0$; a contradiction.\\

Lemma 4: *Let G be a singular graph with adjacency matrix $A = (R_1, R_2, \ldots, R_n)^T$, nullity one and let G have a non-zero row $u_k = (A_{1k}, A_{2k}, \ldots, A_{nk})$ for some $k \in \{1, 2, \ldots, n\}$ in its adjoint. Then a non-trivial kernel relation of the graph is $A_{1k}R_1 + A_{2k}R_2 + \ldots + A_{nk}R_n = 0$.*

Proof: Since $\forall c \in \mathbb{R}$,

$$cA_{1k}a_{i1} + cA_{2k}a_{i2} + cA_{3k}a_{i3} + \ldots + cA_{nk}a_{in} = \begin{cases} 0, & k \neq i \\ \mathrm{Det}(A), & k = i \end{cases} \quad (3)$$

and $\mathrm{Det}(A) = 0$, then $A.(cA_{1k}, cA_{2k}, cA_{3k}, \ldots, cA_{nk})^T = (0, 0, \ldots, 0)^T$, $\forall k \in \{1, 2, \ldots, n\}$. Since u_k is a non-zero row vector, u_k^T, is a kernel eigenvector. Thus a non-trivial kernel relation of the graph is $A_{1k}R_1 + A_{2k}R_2 + \ldots + A_{nk}R_n = 0$, for some $k \in \{1, 2, \ldots, n\}$.\\

Thus the only kernel relation of graph S_2 is $2R_1 + R_2 - R_3 - R_4 - 2R_9 = 0$, obtained using a non-zero row of the adjoint of its adjacency matrix.

Lemma 5: *Let G be a singular graph with nullity one, having the principal kernel eigenvector $v_0 = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T$, where $\alpha_i \neq 0$, for $1 \leq i \leq t,\ 1 \leq t \leq n$, $\alpha_1 > 0$. Then*

i) $\exists i, j \in \{1, 2, \ldots, n\}$ such that $A_{i,j} \neq 0$.

ii) A non-zero row vector of the adjoint is of the form $c(\alpha_1, \alpha_2, \ldots, \alpha_n)$ such that $c \in \mathbb{Z}, c \neq 0$.

Proof:

i) Since $\eta(G) = 1$, the rank of $\mathrm{Adj}(A)$ is one. Thus $\mathrm{Adj}(A)$ has a non trivial row vector u_j which is a kernel eigenvector of A. Thus at least one entry of this row vector which is a cofactor of A is non-zero.

ii) The vector $v_0 = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T$ is the only linearly independent non-zero kernel eigenvector in the one dimensional nullspace of A. Also $\mathrm{Adj}(A)$ has at least one non-zero row vector which is a kernel eigenvector of A. It follows that a non-zero row vector $u_j = (A_{j1}, A_{j2}, \ldots, A_{jn}),\ A_{jk} \in \mathbb{Z},\ \forall k$, is a scalar multiple of v_0. Thus $u_j = cv_o$. This means that $A_{ji} = c\alpha_i,\ \forall i \in \{1, 2, \ldots, n\}$ so that $c \in \mathbb{Q}$. Let $c = \frac{p}{q}$, (in its lowest terms), $p, q \in \mathbb{Z}$. Then q divides each α_i. Hence $q = 1$ and c is an integer. Thus v_0^T is a non-zero row vector of the adjoint (up to integral multiples).\\

Thus since the kernel relation of graph S_3 is $2R_1 + R_2 - R_3 - R_4 = 0$, the row vectors of $\mathrm{Adj}(A)$ are $c(2, 1, -1, -1, 0, 0, 0)$ where $c = -2, -1, 1, 1, 0, 0, 0$, respectively.

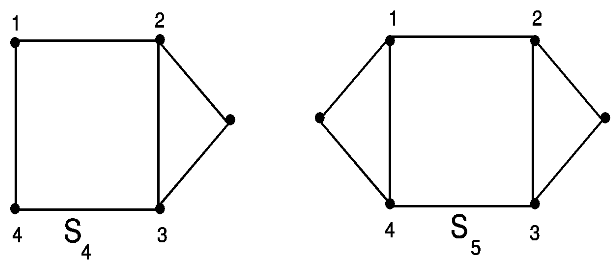

Figure 3:

Graphs S_4 and S_5 have nullity one and the same kernel relation
$R_1 + R_2 - R_3 - R_4 = 0$. The row vectors of $\mathrm{Adj}(A)$ are $g(1, 1, -1, -1, 0)$ with
$g = 1, 1, -1, -1, 0$ and $h(1, 1, -1, -1, 0, 0)$ with $h = -2, -2, 2, 2, 0, 0$ respectively.

Theorem 1: *Let G be a singular graph with nullity one having the principal kernel eigenvector $v_0 = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T$, where $\alpha_i \neq 0$, for $1 \leq i \leq t$, $1 \leq t \leq n$, $\alpha_1 > 0$. Then*

i) $A_{ij} \neq 0,\ \forall i, j \in \{1, 2, \ldots, n\}$.

ii) $\exists c_1, c_2, \ldots, c_n, \in \mathbb{Z}, c_i, \neq 0$ *such that* $\{c_i(\alpha_1, \alpha_2, \ldots, \alpha_n) : i \in \{1, 2, \ldots, n\}\}$ *is the set of row vectors of the adjoint.*

iii) The diagonal entries of the adjoint are of the same sign and $A_{11} : A_{22} : \ldots : A_{nn} = {c_1}^2 : {c_2}^2 : \ldots : {c_n}^2$.

Proof:

i) From Lemma 5 at least one cofactor A_{ij} is not zero. As $\eta(G) = 1$, $r(\mathrm{Adj}(A)) = 1$ and $\exists c \in \mathbb{Z}, c \neq 0$ such that $c(\alpha_1, \alpha_2, \ldots, \alpha_n)$, is a non-zero row vector u_i of the adjoint; $u_i = (A_{i1}, A_{i2}, \ldots, A_{in})$. Since $\alpha_i \neq 0, \forall i$, each cofactor of row i, A_{ij}, $j \in \{1, 2, \ldots, n\}$ is a non-zero integer. Since A is symmetric, so is its adjoint. Hence each row vector u_j of $\mathrm{Adj}(A)$ has non-zero entries. For all $j \in \{1, 2, \ldots, n\}$, $\exists c_j \in \mathbb{Z} - \{0\}$ such that $u_j = (A_{j1}, A_{j2}, \ldots, A_{jn}) = c_j(\alpha_1, \alpha_2, \ldots, \alpha_n)$. So each entry in $\mathrm{Adj}(A)$ (i.e. each cofactor of A) is non-zero.

ii) As each row vector of the adjoint is non-zero and is in the one-dimensional nullspace of A it follows that each row is an integral multiple of the only kernel eigenvector of A, viz.,$(\alpha_1, \alpha_2, \ldots, \alpha_n)$.

iii) In particular, as $A_{ij} \in \mathbb{Z}$ each row vector of the adjoint is a rational multiple of the first row vector of the adjoint. Let the ith row of the adjoint of A be $d_i(A_{11}, A_{12}, \ldots, A_{nn})$, for some $d_i \in \mathbb{Q}$, where from (i) $d_i = c_i/c_1$. Then $A_{jj} = d_j A_{1j}$ and as the adjoint is symmetrical $A_{1j} = d_j A_{11}$. Thus $A_{jj} = d_j^2 A_{11}$. It follows that the diagonal entries of the adjoint are of the same sign and that $A_{11} : A_{22} : \ldots : A_{nn} = 1 : {d_2}^2 : \ldots : {d_n}^2$. The result follows.\\

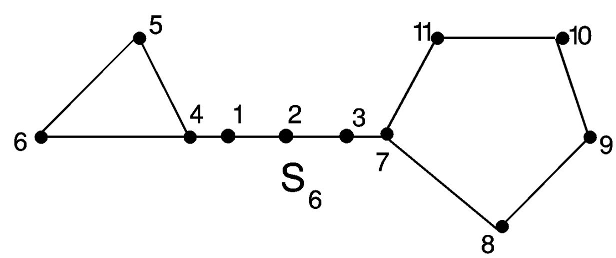

Fig. 4

Graph S_6 is of nullity one and has a kernel relation $-2R_1 + R_2 + 2R_3 - R_4 + R_5 + R_6 - R_7 - R_8 + R_9 + R_{10} - R_{11} = 0$, involving each row vector of A. Thus $A_{11} : A_{22} : \ldots : A_{nn} = 4 : 1 : 4 : 1 : 1 : 1 : 1 : 1 : 1 : 1 : 1$, for $n = 11$.

Lemma 6: *Each diagonal entry of the adjoint of a singular graph with one zero eigenvalue is given by the determinant of the adjacency matrix of the corresponding vertex-deleted subgraph.*

Proof: This follows from the definition of the cofactor A_{ii} obtained by deleting the ith row and the ith column of $A(G)$.\\

Lemma 7: *Let G be a graph of order n and let A be its adjacency matrix. If $\phi(G, \lambda)$ =Det$(\lambda I - A)$, is the characteristic polynomial of G, then $\mathcal{L}$, the coefficient of λ in $\phi(G, \lambda)$, is given by* $\mathcal{L} = (-1)^{n-1}$ (trace of Adj(A)) .

Proof: This follows from the definition of the determinant. We also recall that

$\phi'(G,\lambda) = \sum_{v_i \in \mathcal{V}(G)}^{n} (\phi(G-v_i),\lambda)$. Thus if $\phi(G,\lambda) = \lambda^n + \ldots + \mathcal{L}\lambda + (-1)^n \mathrm{Det}(A(G))$, then
$\phi'(G,\lambda) = n\lambda^{n-1} + \ldots + \mathcal{L}$. It follows that
$\phi'(G,0) = \mathcal{L} = \sum_{v_i \in \mathcal{V}(G)}^{n} (\phi(G-v_i),0) = (-1)^{n-1}\left(\sum_{i=1}^{n} \mathrm{Det}(G-v_i)\right)$.

Thus the coefficient of λ, $\mathcal{L}$, is equal to $(-1)^{n-1}\left(\sum_{i=1}^{n} \mathrm{Det}(G-v_i)\right)$ which is $(-1)^{n-1}(A_{11} + A_{22} + \ldots + A_{nn})$.\\

Thus

i) when $\eta(G) > 1$, as for $G = S_1$, $\mathrm{Trace}(\mathrm{Adj}(A)) = 0$ and so is the coefficient of λ in $\phi(S_1,\lambda) = \mathrm{Det}(\lambda I - A)$, the characteristic polynomial of S_1;

ii) in $G = S_6$ where $\eta(G) = 1$, $\mathrm{Trace}(\mathrm{Adj}(A)) = -17$ and so is the coefficient of λ in the characteristic polynomial of $S_6, viz., \phi(S_6,\lambda) = \mathrm{Det}(\lambda I - A)$.

3 Structure of a Singular Graph

Definition: *Let v_0 be a kernel eigenvector of a singular graph G. A subgraph of G induced by the vertices corresponding to the non-zero components of v_0 is said to be a* **core** *$\chi_{v_0}(=\chi)$ (w.r.t v_0). The number of vertices of the core is called the* **core-order**.

It follows that a core χ (w.r.t v_0) of a singular graph G is a (vertex induced) subgraph of G which is itself singular and has a vector in its nullspace each of whose entries is non-zero. The set of vertices $\mathcal{V}(G\backslash\chi)$ is called the **periphery** of G and is denoted by $\mathcal{P}$.

Definition: *A singular graph Γ of order $n \geq 3$, having a core χ_t and periphery $\mathcal{P} := \mathcal{V}(\Gamma) - \mathcal{V}(\chi_t)$ is a* **minimal configuration,** *of core-number t, if the following conditions are satisfied:*

(i) $\eta(\Gamma) = 1$,

(ii) $\mathcal{P} = \phi$ *or* $\mathcal{P}$ *induces a null graph,*

(iii) *and in the case when* $\mathcal{P} \neq \phi$, *the deletion of* $v \in \mathcal{P}$ *increases the nullity of* Γ.

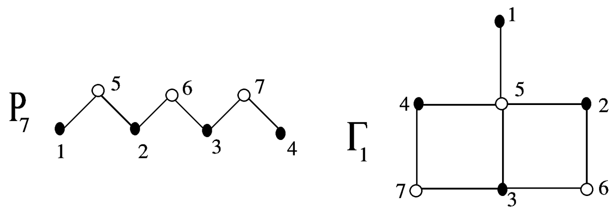

Figure 4:

Thus P_7 and Γ_1 are two minimal configurations for the same core N_4 and the same kernel eigenvector

$$v_0 = (1, -1, 1, -1, 0, 0, 0)^T. \tag{4}$$

The graph Γ_1 has P_7 as a subgraph but both Γ_1 and P_7 are minimal configurations as each has nullity one, $< \mathcal{P} >$ is the null graph N_3, and the deletion of a vertex from the periphery increases the nullity in the resulting graph in each case.

Lemma 8: *Let* (Γ, χ_{v_0}) *be a minimal configuration having the principal kernel eigenvector* $v_0 = (\alpha_1, \alpha_2, \ldots, \alpha_t, 0, \ldots, 0)^T,\ \alpha_i \neq 0,\ 2 \leq t \leq n$. *Then*

i) $\exists c_1, c_2, \ldots, c_t, \in \mathbb{Z}, c_i, \neq 0$ *such that* $\{c_i(\alpha_1, \alpha_2, \ldots, \alpha_t, 0, 0, \ldots, 0) : i \in \{1, 2, \ldots, t\}\}$ *is the set of row vectors of the adjoint corresponding to the vertices of the core.*

ii) *For* $i > t$ *the cofactors* $A_{ik} = 0,\ k \in \{1, 2, \ldots, n\}$.

iii) *The diagonal entries of the adjoint are in the ratio* $A_{11} : A_{22} : \ldots : A_{nn} = {c_1}^2 : {c_2}^2 : \ldots : {c_t}^2 : 0 : 0 : \ldots : 0.$

iv) $c_1 : c_2 : \ldots : c_t = \alpha_1 : \alpha_2 : \ldots : \alpha_t$ *and* $A_{11} : A_{22} : \ldots : A_{nn} = \alpha_1^2 : \alpha_2^2 : \ldots : \alpha_t^2 : 0 : 0 : \ldots : 0.$

Proof: By Lemma 5, the vector $w = c(\alpha_1, \alpha_2, \ldots, \alpha_t, 0, 0, \ldots, 0)^T,\ c \in \mathbb{Z} - \{0\}$ is an eigenvector in the one-dimensional nullspace of A and w^T is also a row vector of $\mathrm{adj}(A)$, for particular integral values of c. Since $r(\mathrm{Adj}(A)) = 1$, and $\mathrm{Adj}(A)$ is symmetric, then each of the first t rows is non-zero and result (i) follows.

Also for $k \in \{1, 2, \ldots, t\}$ & $i > t$ each entry A_{ki} in a row vector of the adjoint is 0 so that the cofactors $A_{ki} = 0$, and result (ii) now follows. In particular $A_{jj} = 0, j > t$. By a proof similar to that for Theorem 1 part (iii), result (iii) now follows. Results (ii) and (iii) agree also with the definition of a minimal configuration Γ, since if a vertex v of the periphery is deleted, then the nullity of the subgraph obtained is more than one. Thus $r(\Gamma - v) < n - 1$ and each cofactor of $A(\Gamma - v)$ is zero. In particular, $A_{jj} = 0, j > t$.

The diagonal entries of the adjoint are $c_1\alpha_1, c_2\alpha_2, \ldots, c_t\alpha_t, 0, 0, \ldots, 0,\ c_i \in \mathbb{Z}$. As $A_{ii} = c_i\alpha_i$, then from (iii) $c_1 : c_2 : \ldots : c_t = \alpha_1 : \alpha_2 : \ldots : \alpha_t$. Applying (iii) again, the proof of result (iv) is now complete.\\

Thus the adjoint of the adjacency matrix of the minimal configuration Γ_1, of Fig. 5, is

$$\begin{pmatrix} -1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
, and satisfies the results of Lemma 8.\\

Theorem 2: *Let (Γ, χ_{v_0}) be a minimal configuration having core χ_t and the principal kernel eigenvector $v_0 = (\alpha_1, \alpha_2, \ldots, \alpha_t, 0, \ldots, 0)^T$ where $\alpha_i \neq 0$, for $1 \leq i \leq t$, $1 \leq t \leq n$, $\alpha_1 > 0$. Let A be the adjacency matrix of Γ and $\phi(\Gamma, \lambda) = Det(\lambda I - A)$, be the characteristic polynomial of Γ. Then $\exists \nu \in \mathbb{Z} - \{0\}$ such that $\mathcal{L}$, the coefficient of λ in $\phi(\Gamma, \lambda)$, is equal to $\nu \left({\alpha_1}^2 + {\alpha_2}^2 + \ldots + {\alpha_t}^2\right)$; i.e. the squared norm of v_0 divides $\mathcal{L}$.*

Proof: Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} \end{pmatrix}. \tag{5}$$

In block form,

$$A = \left(\begin{array}{c|c} \chi_t & P \\ \hline P^T & <\mathcal{P}> \end{array} \right), \tag{6}$$

where $\mathcal{P}$ is the periphery and P describes the edges between the core and the periphery. By Lemma 7, $\mathcal{L}$, the coefficient of λ in the characteristic polynomial of Γ, viz., $\phi(\Gamma,\lambda) = \mathrm{Det}(\lambda I - A)$, is equal to $(-1)^{n-1}\sum_{i=1}^{n}\mathrm{Det}(\Gamma - v_i)$ which is $(-1)^{n-1}(A_{11} + A_{22} + \ldots + A_{nn})$. As by definition of minimal configuration, $\eta(\Gamma) = 1$, then $r(\mathrm{Adj}(A)) = 1$ and $(\alpha_1, \alpha_2, \ldots, \alpha_t, 0, 0, \ldots, 0)^T$ is the unique linearly independent kernel eigenvector in the nullspace of A. Then by Lemmas 4 and 8,
$\forall k \in \{1, 2, \ldots, t\},\ \exists c_k \in \mathbb{Z}-\{0\}$ such that $(\alpha_1, \alpha_2, \ldots, \alpha_t) = (A_{1k}/c_k, A_{2k}/c_k, A_{3k}/c_k, \ldots, A_{tk}/c_k)$.

Hence each row vector of the adjoint is a rational multiple of the first row vector of the adjoint. Let the ith row of the adjoint of A be $d_i(A_{11}, A_{12}, \ldots, A_{1n})$, for some $d_i \in \mathbb{Q}$, where $d_i = c_i/c_1$. Thus the adjoint

$$\mathrm{Adj(A)} = \begin{pmatrix} A_{11} & d_2A_{11} & d_3A_{11} & \ldots & d_tA_{11} & 0 & \ldots & 0 \\ d_2A_{11} & (d_2)^2A_{11} & d_2d_3A_{11} & \ldots & d_2d_tA_{11} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ d_tA_{11} & d_td_2A_{11} & d_td_3A_{11} & \ldots & (d_t)^2A_{11} & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \end{pmatrix}. \tag{7}$$

The trace of the adjoint

$$\begin{aligned} \mathrm{tr}(\mathrm{Adj}(A)) &= \sum_{i=1}^{t}\mathrm{Det}(G - v_i) \\ &= A_{11}(1 + d_2^2 + d_3^2 + \ldots + d_t^2) \end{aligned}$$

and $\mathrm{tr}(\mathrm{Adj}(A)) = (-1)^{n-1}\mathcal{L}$, by Lemma 7.

As

$$\begin{aligned} A_{11} : A_{12} : \ldots : A_{1t} &= 1 : d_2 : \ldots : d_t \\ &= c_1 : c_2 : \ldots : c_t \\ &= \alpha_1 : \alpha_2 : \ldots : \alpha_t, \text{by Lemma 8iv} \end{aligned}$$

and $A_{ik} = c_i\alpha_k = c_k\alpha_i,\ k, i \in \{1, 2, \ldots, t\}$, then $|\mathcal{L}| = \left|\frac{c_1}{\alpha_1}\right| (\alpha_1^2 + \alpha_2^2 + \ldots + \alpha_t^2) = |v|^2$, where v is the kernel eigenvector $\sqrt{\nu} v_0 = \sqrt{\left|\frac{c_1}{\alpha_1}\right|}(\alpha_1, \alpha_2, \ldots, \alpha_t, 0, \ldots, 0)^T$.
Also since
$(c_1, c_2, \ldots, c_t) = \nu(\alpha_1, \alpha_2, \ldots, \alpha_t)$ and $(\alpha_1, \alpha_2, \ldots, \alpha_t, 0, \ldots, 0)^T$ is the principal kernel eigenvector of Γ, then ν $(= \frac{c_1}{\alpha_1} = \frac{A_{11}}{{\alpha_1}^2})$ is an integer as required.\\

Thus graphs S_3 and S_5 are minimal configurations with $|\mathcal{L}|$ equal to 7 and 4 respectively and kernel eigenvectors v_0 equal to $(-2, -1, 1, 1, 0, 0, 0)^T$ and $(1, 1, -1, -1, 0)^T$ respectively. In each case $|\mathcal{L}| = |v_0|^2$.

The following table gives the characteristic polynomials of the minimal configurations, of order 7, whose core is the disjoint union of C_4 and N_1 and whose adjacency matrix satisfies the kernel relation

$$R_1 + R_2 - R_3 - R_4 + R_5 = 0. \tag{8}$$

Table 1

The coefficient of λ in $\phi(\Gamma, \lambda)$,viz., $\mathcal{L}$, is equal to $(-1)^{n-1} \left(\sum_{i=1}^{n} \mathrm{Det}(G - v_i) \right)$ which is
$(-1)^{n-1} (A_{11} + A_{22} + \ldots + A_{nn})$ by Lemma 7. In the set of minimal configurations, given in Table 1, it is noticed that $\mathcal{L}$ is -5, numerically equal to the square of the norm of a kernel eigenvector. Another minimal configuration with $|\mathcal{L}| = 5$ is P_9 (the path on 9 vertices). The characteristic polynomial of P_9 is $\lambda^9 - 8\lambda^7 + 21\lambda^5 - 20\lambda^3 + 5\lambda$. The core in this case is N_4, the kernel relation is (8) above and $\mathcal{L}$ is 5.

Table 2 gives $\mathcal{L}$ for the graphs mentioned in this paper.

Graph	*Order*	*Nullity*	$\mathcal{L}$	$\mathcal{R}$
S_1	8	2	0	$R_1 - R_2 + R_3 - R_4 = 0,\ R_6 + R_7 - R_5 = 0$
S_2	9	1	11	$-2R_1 - R_2 + R_3 + R_4 + 2R_9 = 0$
S_3	7	1	-7	$-2R_1 - R_2 + R_3 + R_4 = 0$
S_4	5	1	4	$R_1 + R_2 - R_3 - R_4 = 0$
S_5	6	1	8	$R_1 + R_2 - R_3 - R_4 = 0$
S_6	11	1	-17	$-2R_1 + R_2 + 2R_3 - R_4 + R_5$
				$+R_6 - R_7 - R_8 + R_9 + R_{10} - R_{11} = 0$
S_7	15	1	-7	$2R_1 + R_2 - R_3 - R_4 = 0$
S_8	13	1	7	$2R_1 + R_2 - R_3 - R_4 = 0$

Table 2.

It is noted that although S_5 has nullity one, it is not a minimal configuration. The value of $\mathcal{L}$ is 8 which is not equal to the square of the norm of the principal kernel eigenvector. In view of the above data the following conjecture is formulated.

Conjecture: *Minimal configurations with the same core and kernel relation have the same value of $\mathcal{L}$, numerically equal to the square of the norm of the kernel eigenvector with integer coefficients whose g.c.d. is one.*

Theorem 2 may be extended, with the same proof, to any singular graph of nullity one as follows.

Theorem 3: *Let G be a singular graph with nullity one and principal kernel eigenvector $v_0 = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T$, where $\alpha_i \neq 0$, for $1 \leq i \leq t,\ 1 \leq t \leq n$, $\alpha_1 > 0$. Let G have core χ_t. Let A be the adjacency matrix of G and $\phi(G, \lambda) = Det(\lambda I - A)$ be the characteristic polynomial of G. Then $\mathcal{L}$, the coefficient of λ in $\phi(G, \lambda)$ is numerically equal to the square of the norm of a kernel eigenvector $\sqrt{\nu}(\alpha_1, \alpha_2, \ldots, \alpha_t, 0, \ldots, 0)^T$ for a particular $\nu \in \mathbb{Z}, \nu \neq 0$.*
\\\\

4 The Deck of Vertex-deleted Subgraphs

Theorem 4: *Let G be a singular graph of nullity one, with core χ and periphery $\mathcal{P}$. Then the vertex-deleted subgraphs corresponding to the vertices of χ are non-singular and those corresponding to the vertices of $\mathcal{P}$ are singular.*

Proof: Since $\eta(G) = 1$ then by Lemma 3, $r(\mathrm{Adj}(A)) = 1$, where A is the adjacency matrix of G. By Lemma 4, the unique non-trivial kernel relation of the graph G is
$A_{1k}R_1 + A_{2k}R_2 + \ldots + A_{nk}R_n = 0$, for some $k \in \{1, 2, \ldots, n\}$, where u_k is any non-zero row vector $(A_{1k}, A_{2k}, \ldots, A_{nk})$ in the adjoint of $A(G)$. The vertices of the core are labelled $v_1, v_2, \ldots, v_t$, and, as for a minimal configuration, since $\mathrm{Adj}(A)$ is symmetric, for $i, k \leq t$, $A_{ik} \neq 0$, whereas all other cofactors in $\mathrm{Adj}(A)$ are zero. Thus $A_{ii} \neq 0$ for $i \leq t$ and $A_{ii} = 0$ for $i > t$. By Lemma 6, A_{ii} gives the determinant of the corresponding vertex-deleted subgraph $G - v_i$. Only for adjacency matrices of the vertex-deleted subgraphs corresponding to the vertices of the core are the determinants not zero. Thus the only vertex deleted subgraphs which are singular are those corresponding to the periphery.\\

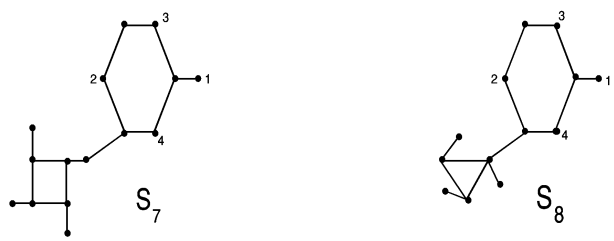

Figure 5:

It can be verified that graphs S_7 and S_8 are of nullity one and only the vertex deleted subgraphs corresponding to the core are non-singular.

Corollary: *Let G be a singular graph without isolated vertices, having adjacency matrix A. Then G has nullity one if and only if there are at least two non-singular graphs in its deck of vertex-deleted subgraphs.*

Proof: If G has nullity one then by Theorem 4, the vertex-deleted subgraphs corresponding to the core are non-singular. As for graphs without isolated vertices the smallest core is N_2 [7], the result follows.

Conversely, if G is singular then $r(\mathrm{Adj}(A)) = 1$ when $\eta(G) = 1$, and $\mathrm{Adj}(A) = 0$ when $\eta(G) > 1$. If there are at least two non-singular graphs in the deck of the vertex-deleted subgraphs of G then $\mathrm{Adj}(A) \neq 0$. Thus $\eta(G) = 1$.\\

Corollary: *Let Γ be a minimal configuration. Then every singular vertex-*

deleted subgraph of Γ *has nullity equal to* 2.

Proof: Since Γ is a minimal configuration, $\eta(\Gamma) = 1$. If a vertex-deleted subgraph $\Gamma - v_i$, where $v_i \in \mathcal{V}(\Gamma)$ is singular then by Theorem 4, $v_i \in \mathcal{P}$. By definition of minimal configuration the deletion of a vertex from the periphery increases the nullity of G. Hence $\eta(\Gamma - v_i) \geq 2$. Also, by the interlacing theorem, $\eta(\Gamma - v_i) \leq 2, \forall i$. Hence for minimal configurations, the singular vertex-deleted subgraphs of Γ correspond to the periphery and each has nullity equal to 2. \\

The converse is false as can be seen by considering the graph S_3 with an edge joining 2 vertices of the periphery, which is not a minimal configuration. Its vertex-deleted subgraphs which are singular have nullity 2. It is noted that not all minimal configurations have a periphery so that singular vertex-deleted subgraphs need not appear [8].

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